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# A VARIATION OF THE TCHEBYCHEFF QUADRATURE PROBLEM ON THE INFINITE INTERVAL

by

Wayne Allan Walker

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled A VARIATION OF THE TCHEBYCHEFF QUADRATURE PROBLEM ON THE INFINITE INTERVAL submitted by Wayne Allan Walker in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis is concerned with the degree of exactness of Tchebycheff and modified Tchebycheff quadrature formulae on the semi-infinite and infinite intervals.

The first Chapter is a historical survey, giving an outline of the development of the theory of Tchebycheff quadrature and indicating some of the areas of recent research. In Chapter 2 we examine a modified Tchebycheff quadrature formula with $n$ nodes on the interval $(-\infty,\infty)$ and show that the degree of exactness of this formula is of order $\log n$. The result is an extension of the results of Erdös and Sharma, and Meir and Sharma to the infinite interval, although the modified Tchebycheff formula is less general than theirs.

Chapter 3 deals with Tchebycheff quadrature on the infinite and semi-infinite intervals with weight function $e^{-x^2}|x|^{2\alpha}$, $\alpha > -\frac{1}{2}$ for $(-\infty,\infty)$ and the weight $e^{-x}x^{\alpha}$, $\alpha > -1$ for $(0,\infty)$. These results are qualitative extensions of the results of Salzer and Gatteschi. In Chapter 4, a partial result is given concerning the degree of exactness of a modified Tchebycheff quadrature formula on the interval $(0,\infty)$ and a conjecture is formulated.

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# CHAPTER I

# INTRODUCTION - A HISTORICAL SURVEY

## 1.1 GAUSS QUADRATURE FORMULA

The problem of quadrature or of finding the area inside a given closed curve has attracted the attention of mathematicians since the time of Archimedes, when he found the area of a circle by subdividing it into small triangles. The essential step in almost all quadrature formulae, at least those of interpolatory type, seems to be to replace a given function by a simpler function. Thus in the trapezoidal formula, we replace a given function by a broken line which interpolates the function in a certain number of points, while in Simpson's formula, a parabolic arc replaces the given function over certain intervals. The same idea underlies the general Newton-Cotes formula, where the function is replaced by a suitable polynomial which interpolates the function in equidistant points.

Among the plethora of quadrature formulae available today, the Gauss formula has an importance of its own. This formula is also of an interpolatory character, but here the nodes of interpolation are not fixed in advance as is done in Simpson's or the Newton-Cotes formula. If

we construct, for any continuous function $f(x)$, the Lagrange interpolation polynomial

$$L(x) = \sum_{k=1}^{n} f(x_k)\, \ell_k(x)$$

where

$$\ell_k(x) = \frac{\lambda(x)}{\lambda'(x_k)(x-x_k)} \; ; \quad \lambda(x) = \prod_{k=1}^{n} (x-x_k)$$

and $x_1 < x_2 < \cdots < x_n$ are arbitrarily assigned nodes, we have $f(x) = L(x) + R_n(x)$ where $R_n(x)$ vanishes whenever $f(x) \varepsilon \Pi_{n-1}$, the class of all polynomials of degree $\leq n-1$. Indeed

$$R_n(x) = \left[\frac{\lambda(x)}{n!}\right] f^n(\xi) \; , \; x_1 < \xi < x_n \; .$$

Hence, whenever $f(x) \varepsilon \Pi_{n-1}$, we have

$$(1.1.1) \qquad \int_{-1}^{1} f(x)\, dx = \sum_{k=1}^{n} A_k\, f(x_k) \; , \quad A_k = \int_{-1}^{1} \ell_k(x)\, dx \; .$$

By appropriately choosing the $n$ nodes of interpolation, it is possible to increase the class of polynomials for which (1.1.1) is exact. The optimum situation is achieved when the $x_k$'s are taken to be the zeros of $P_n(x)$, the Legendre polynomial of degree $n$. This gives the Gauss quadrature formula:

$$(1.1.2) \qquad \int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} A_k f(x_k), \qquad A_k > 0$$

valid for all polynomials of degree $\leq 2n-1$.

Closely related to the Gauss quadrature formula are those due to Radau and Lobatto. If the end point -1, of the interval of integration, is an assigned node, we then have the Radau quadrature formula. Assigning both the end points -1 and 1 as nodes, we obtain the Lobatto quadrature formula.

For a more exhaustive treatment of the theory of approximate integration, we refer to Krylov [9] or Natanson [14].

## 1.2 GENERALIZATIONS OF GAUSS QUADRATURE

Turán [21] has generalized the Gauss quadrature formula by considering derivatives of the function in the quadrature formula. The existence of a formula of the type:

$$(1.2.1) \qquad \int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} f(x_k)\lambda_k^{(0)} + \sum_{k=1}^{n} f'(x_k)\lambda_k^{(1)}$$

follows immediately from the interpolation formula

$$f(x) = \sum_{k=1}^{n} f(x_k)\,\ell_{k,0}(x) + \sum_{k=1}^{n} f'(x_k)\,\ell_{k,1}(x)$$

valid for all polynomials of degree $\leq 2n-1$, where, using the notation of the Lagrange interpolation polynomial,

$$\ell_{k,0}(x) = \left[1 - \frac{\lambda''(x_k)}{\lambda'(x_k)}(x-x_k)\right] \ell_k^2(x)$$

$$\ell_{k,1}(x) = (x-x_k)\, \ell_k^2(x) .$$

For any choice of $x_k$, it is only possible to have (1.2.1) valid for polynomials of degree $\leq 2n-1$.

Turán has given a generalization of formula (1.2.1) in the form

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=1}^{n} [f(x_k)\lambda_k^{(0)} + f'(x_k)\lambda_k^{(1)} + \cdots + f^{(m_k-1)}(x_k)\lambda_k^{(m_k-1)}]$$

valid for all polynomials of degree $\leq(m_1+m_2+\cdots+m_n-1)$. If $m_1 = m_2 = \cdots = m_n = k$, Turán's formula is valid for all polynomials of degree $\leq kn-1$ and may be made valid for all polynomials of degree $\leq(k+1)n-1$, when $k$ is odd, by a suitable choice of the nodes.

Considering this type of quadrature formula with weight function $(1-x^2)^{-\frac{1}{2}}$, Kiš [8] has given, for $k=3$, the explicit form of the quadrature formula considered by Turán which is valid for all polynomials of degree $\leq 4n-1$

$$\int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{n} \left\{ \sum_{k=1}^{n} g(\cos \frac{2k-1}{2n} \pi) \right.$$

$$- \frac{1}{4n^2} \sum_{k=1}^{n} \cos \frac{2k-1}{2n} \pi\, g'(\cos \frac{2k-1}{2n} \pi)$$

$$\left. + \frac{1}{4n^2} \sum_{k=1}^{n} \sin^2 \frac{2k-1}{2n} \pi\, g''(\cos \frac{2k-1}{2n} \pi) \right\} .$$

This is a generalization of the Hermite formula of quadrature (formula (1.4.1)).

## 1.3 ERROR ESTIMATES

If we are applying a quadrature formula, it is of interest to have some estimate of the error involved. For an arbitrary quadrature formula

$$\int_a^b f(x)\, dx = L(f) \equiv \sum_{k=0}^{m-1} A_k\, f(x_k)$$

which is exact for polynomials of degree $\leq r-1$, a natural question arises regarding the error when $f(x)$ belongs to a class of functions bigger than $\Pi_{r-1}$. A discussion of the various aspects of the error estimates for certain classes of functions and the choice of nodes which would minimize the error in some norm would lead us too far afield. We refer here to the works of Nikol'skii [15] and Sard [17].

## 1.4 TCHEBYCHEFF QUADRATURE

In 1867, Hermite gave the formula

$$(1.4.1) \qquad \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{n} \sum_{k=1}^{n} f(\cos \frac{2k-1}{2n}\pi)$$

which is exact for all polynomials of degree $\leq 2n-1$ and so is a Gauss-type formula. Tchebycheff noted the advantage of this type of formula in numerical computation and in 1873 posed the following problem:

Find a real $A$ and real nodes $x_k (1 \leq k \leq n)$* in such a way that

$$(1.4.2) \qquad \int_{-1}^{1} f(x)\, w(x)\, dx = A \sum_{k=1}^{n} f(x_k)$$

is exact for all polynomials $f(x)$ of degree $\leq n$. It is only necessary to consider the interval of integration to be $[-1,1]$ since any finite interval $[a,b]$ can be reduced to $[-1,1]$ by the linear transformation $\frac{2x-(a+b)}{b-a} \to x$. A Tchebycheff quadrature formula is said to be possible for $n$ with weight function $w(x)$ if there exist numbers $x_k (k=1,2,\ldots,n)$ lying interior to $[-1,1]$ such that (1.4.2) is true for every polynomial of degree $\leq n$.

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* It should be emphasized that $x_k \equiv x_k^{(n)}$ and $A \equiv A^{(n)}$ depend on $n$, but for typographical reasons we shall here and throughout the thesis omit the superscript $n$ with the understanding that this dependence on $n$ is implied in the formulation of the problems and the statements of the results.

In connection with the formula of Hermite, it is natural to ask if there exist weight functions other than $(1-x^2)^{-\frac{1}{2}}$ for which a Gauss formula of Tchebycheff type exists, i.e. which has equal weights and which is exact for polynomials of degree $\leq 2n-1$. This question was answered by K.A. Posse and N.A. Sonin and the result is contained in the following:

<u>Theorem</u> (Krylov [9], p. 184). If for arbitrary values of $n = 1,2,\ldots$ there exist constants $A$ with $x_k$ the zeros of the polynomial orthogonal to $w(x)$ on [-1,1], such that (1.4.2) is exact for $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$, then $w(x)$ coincides with the Tchebycheff weight function $(1-x^2)^{-\frac{1}{2}}$.

A natural method of determining the nodes of a Tchebycheff quadrature formula is to set $f(x) = x^k$ $(k=1,2,\ldots,n)$ and the resulting system of simultaneous equations will yield solutions which are the desired nodes. Setting $w(x) = 1$ in (1.4.2), a direct computation shows that the resulting systems have real solutions contained in [-1,1] for $n \leq 7$.

## 1.5 <u>EXPLICIT FORMULAE FOR THE NODES IN TCHEBYCHEFF QUADRATURE</u>

Tchebycheff chose an analytical approach in determining the nodes of the quadrature formula (1.4.2). Choosing $f(x) = 1$, we are able to determine the weight

$$A = \frac{\lambda}{n} \quad ; \quad \lambda = \int_{-1}^{1} w(x)\, dx \ .$$

Following Tchebycheff's method, we choose $f(x) = \frac{1}{z-x}$ and employ (1.4.2) to yield the remainder

$$(1.5.1) \qquad R\left(\frac{1}{z-x}\right) = \int_{-1}^{1} \frac{w(x)}{z-x}\,dx - \frac{\lambda}{n}\sum_{k=1}^{n} \frac{1}{z-x_k}$$

$$= \int_{-1}^{1} \frac{w(x)}{z-x}\,dx - \frac{\lambda}{n}\,\frac{\Pi'(z)}{\Pi(z)}$$

where

$$\Pi(z) = (z-x_1)(z-x_2)\,\ldots\,(z-x_n)\,.$$

Choosing $z$ such that $|z|>1$, we can write $f(x)$ as

$$f(x) = \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{z^{\nu+1}}\,.$$

We can also write

$$\frac{\Pi'(z)}{\Pi(z)} = \sum_{\nu=0}^{\infty} \frac{S_{\nu}}{z^{\nu+1}}$$

where

$$S_{\nu} = \sum_{k=1}^{n} x_k^{\nu}\,.$$

We can then express the remainder of (1.4.2) as

$$R\left(\frac{1}{z-x}\right) = \sum_{\nu=0}^{\infty} \frac{\mu_{\nu} - \frac{\lambda}{n} S_{\nu}}{z^{\nu+1}}$$

where

$$\mu_{\nu} = \int_{-1}^{1} w(x)\, x^{\nu}\,dx\,.$$

Assuming the quadrature formula to be valid for all polynomials $\varepsilon\ \Pi_n$

$$\mu_\nu - \frac{\lambda}{n} S_\nu = 0 \qquad \nu = 0,1,\ldots,n\ .$$

We can then write (1.5.1) as

$$\int_{-1}^{1} \frac{w(x)}{z-x}\,dx - \frac{\lambda}{n}\frac{\Pi'(z)}{\Pi(z)} = \sum_{\nu=n+1}^{\infty} \frac{\mu_\nu - \frac{\lambda}{n} S_\nu}{z^{\nu+1}}\ . \tag{1.5.2}$$

Integrating with respect to $z$ and simplifying, we have

$$\Pi(z)\exp\Big[\sum_{\nu=n+1}^{\infty} \frac{S_\nu - \frac{n}{\lambda}\mu_\nu}{\nu\, z^\nu}\Big] = A\exp\Big[\frac{n}{\lambda}\int_{-1}^{1} w(x)\log(z-x)\,dx\Big] \tag{1.5.3}$$

where $A$ is some constant. Since the expansion of $\exp\Big[\sum_{\nu=n+1}^{\infty} \frac{S_\nu - \frac{n}{\lambda}\mu_\nu}{\nu\, z^\nu}\Big]$ in powers of $\frac{1}{z}$ is of the form $1 + \frac{C}{z^{n+1}} + \ldots$, and the expansion of $A\exp\Big[\frac{n}{\lambda}\int_{-1}^{1} w(x)\log(z-x)\,dx\Big]$ is of the form $A z^n + B z^{n-1} + \ldots$, $\Pi(z)$ is given by the polynomial part of the right hand side of (1.5.3).

The following table summarizes the results of Tchebycheff using this method for $w(x) = 1$ and $n \le 7$.

| n | $\Pi(x)$ | nodes |
|---|---|---|
| 1 | $x$ | $0$ |
| 2 | $x^2 - \frac{1}{3}$ | $\pm 0.577350$ |
| 3 | $x^3 - \frac{x}{2}$ | $0, \pm 0.707107$ |
| 4 | $x^4 - \frac{2}{3}x + \frac{1}{45}$ | $\pm 0.794654$, $\pm 0.187592$ |
| 5 | $x^5 - \frac{5}{6}x^3 + \frac{7}{72}x$ | $0, \pm 0.832497$, $\pm 0.374541$ |
| 6 | $x^6 - x^4 + \frac{x^2}{5} - \frac{1}{105}$ | $\pm 0.866247$, $\pm 0.422519$, $\pm 0.266635$ |
| 7 | $x^7 - \frac{7}{6}x^5 + \frac{119}{360}x^3 - \frac{149}{6480}x$ | $0, \pm 0.883862$, $\pm 0.529657$, $\pm 0.323912$ |

It was later shown that for $n = 8$ and $n = 10$, the zeros of the polynomials considered by Tchebycheff were imaginary, and that they are real for $n = 9$.

## 1.6 IDEAS OF BERNSTEIN

Tchebycheff thus left open the problem whether Tchebycheff quadrature is possible for $n \geq 10$. In 1932, Bernstein proved the following:

Theorem: The Tchebycheff problem cannot be solved for $n > 9$.

In other words, the quadrature formula

$$(1.6.1) \qquad \int_{-1}^{1} f(x)\, dx = \frac{2}{n} \sum_{k=1}^{n} f(x_k)$$

does not hold for $n > 9$ with $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$. The proof results from the following lemmas of Bernstein:

Lemma 1. Let the formula (1.6.1) be true for any polynomial of degree $\leq 2m-1$, $m < n$. Then

$$x_1 < \xi_1^{(m)}$$

where $\xi_1^{(m)}$ is the smallest zero of $P_m(x)$, the Legendre polynomial of degree $m$.

Lemma 2. If the Tchebycheff formula (1.6.1) is valid for all polynomials of degree $\leq 2m-1$, $m < n$, then

$$(1.6.2) \qquad \frac{2}{n} < A_1^{(m)}$$

where $A_i^{(m)}$ denote the Christoffel numbers of the Gauss quadrature formula (1.1.2).

Using the estimates (Natanson [14], pp. 522):

$$A_1^{(m)} < \frac{\pi}{m} \sqrt{1-[\xi_1^{(m)}]^2}$$

$$-1 < \xi_1^{(m)} < -1 + \frac{3}{m(m+1)} \qquad (m > 6)$$

in the result of Lemma 2, it is shown that for $n > 9$, the inequality (1.6.2) fails to hold.

## 1.7 ASYMPTOTIC DISTRIBUTION OF TCHEBYCHEFF NODES

Although Bernstein's result established that not all the nodes of the Tchebycheff quadrature formula (1.6.1) will be real for $n > 9$, he did not give any information as to their location. R.O. Kuzmin [10] showed that if, in formula (1.5.3), $w(x) = 1$, then the zeros of $\pi(x)$, which are the nodes of the quadrature formula, are found, with the exception of $x = 0$ for odd $n$, in the neighborhood of $\rho(x) = 1$ as $n \to \infty$ where

$$\rho(x) = \int_{-1}^{1} \log |x-t| \, dt \, .$$

This result was rediscovered by Delange [2] who considered the location of the nodes of the Tchebycheff quadrature formula by examining the asymptotic expansion of $\pi_n(z)$ defined by

$$z^n \exp\left[ -n \sum_{j=1}^{\infty} \frac{1}{2j(2j+1)z^{2j}} \right] = \Pi_n(z) + \sum_{j=1}^{\infty} \frac{\alpha_j^{(n)}}{z^j} .$$

Studying the asymptotic distribution of the zeros of $\Pi_n(z)$ when $n \to \infty$, it is shown that all the zeros, other than the origin for odd $n$, are found arbitrarily close to a curve $C$ having the shape of an ellipse with major axis $[-1,1]$ and minor axis of length .52552.

## 1.8 TCHEBYCHEFF QUADRATURE ON INFINITE AND SEMI-INFINITE INTERVAL (Results of H. Salzer)

Although Tchebycheff originally posed his problem for the finite interval, quadrature formulae on the infinite interval are equally important. H.E. Salzer [16] has considered the Tchebycheff-Hermite quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = A \sum_{i=1}^{n} f(x_i) \tag{1.8.1}$$

and the Tchebycheff-Laguerre quadrature formula

$$\int_{0}^{\infty} e^{-x} f(x)\, dx = A \sum_{i=1}^{n} f(x_i) . \tag{1.8.2}$$

For the Tchebycheff-Laguerre quadrature formula to be valid for polynomials of degree $\leq n$, it is necessary and sufficient to solve the system of equations

$$\sum_{i=1}^{n} x_i^j = n(j!) \qquad j=1,2,\ldots,n\,.$$

The $x_i$ are the zeros of a polynomial

$$\Phi_n(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

where the $a_i$ are defined successively by

$$ja_j + a_{j-1} \sum_{i=1}^{n} x_i + a_{j-2} \sum_{i=1}^{n} x_i^2 + \ldots + a_1 \sum_{i=1}^{n} x_i^{j-1} + \sum_{i=1}^{n} x_i^j = 0$$

$$(j=1,2,\ldots,n).$$

Salzer [16] gives these polynomials explicitly and obtains their zeros for $n \leq 10$. It is seen that for $n=2$, $x_1=0$ and $x_2=2$, and that for $3 \leq n \leq 10$, there exist imaginary zeros.

For the Tchebycheff-Hermite quadrature formula, we are required to solve the system of equations

$$\sum_{i=1}^{n} x_i^j = \left\{ \begin{array}{ll} 0 & \text{for } j \text{ odd} \\[2ex] \dfrac{n.1.3\ldots(j-1)}{2^{j/2}} & \text{for } j \text{ even} \end{array} \right\} \quad j=1,2,\ldots,n\,.$$

As before, Salzer obtains the nodes by determining the polynomial of which they are the zeros and then solves for the zeros. For $n = 2$ we have $x_1 = -\sqrt{\frac{1}{2}}$, $x_2 = \sqrt{\frac{1}{2}}$ and for $n = 3$

$$x_1 = -\sqrt{\frac{3}{2}}\ ,\ x_2 = 0\ ,\ x_3 = \sqrt{\frac{3}{2}}\ .$$

For $3 < n \leq 10$ there exist imaginary nodes and hence Tchebycheff-Hermite quadrature is not possible.

Burgoyne [1] has obtained polynomials using the method of Salzer for $n \leq 50$ and has employed Descartes' rule of signs to establish the maximum number of real zeros in the interval of integration. He concludes that Tchebycheff quadrature of the type considered by Salzer is not possible for $10 < n \leq 50$.

1.9 <u>CONTRIBUTION OF GATTESCHI</u>

The considerations so far have not established whether or not Tchebycheff quadrature may be possible for some large values of $n$. Bernstein's method, as outlined earlier in this chapter, cannot be applied, since as pointed out by Salzer "... his method fails when applied in exactly the same way to prove the impossibility of (1.8.1) or (1.8.2) (for $x_i$'s lying within the interval of integration) for larger $n$. The reason is that it leads to a similar inequality, but now involving the first Christoffel number of the Laguerre or Hermite polynomials, and that inequality seems to always hold for larger values of $n$". In view of the above remark, it is interesting to observe that

Gatteschi [5] has resolved the problem by a method very similar to that of Bernstein. Denoting by $K_n$ the constant of the Tchebycheff formula

$$(1.9.1) \qquad \int_a^b w(x)\, f(x)\, dx = K_n \sum_{k=1}^{n} f(x_k)$$

and $C_{m,1}$ as the first Christoffel number of the Gauss formula

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=1}^{m} C_{m,k}\, f(x_k)\ ,$$

Gatteschi proves the following:

<u>Lemma</u>. If (1.9.1) has a degree of precision $2m-1$, $m < n$, then

$$K_n < C_{m,1} \quad .$$

By the relation (Szegö [19], p. 360)

$$C_{m,i} = \frac{1}{\lambda_{m,i}[L_m'(\lambda_{m,i})]^2}$$

where $\lambda_{m,i}$ are the zeros of $L_m(x)$ .

Similarly in the Tchebycheff-Hermite case we have (Szegö [19], p. 360)

$$C_{m,i} = \frac{\sqrt{\pi}\, 2^{2m+1}\, m!}{[H_m'(h_{m,i})]^2}$$

where $h_{m,i}$ are the zeros of $H_m(x)$ .

Establishing lower bounds on $L_m'(\lambda_{m,1})$ and $H_m'(h_{m,1})$, Gatteschi employes the preceeding Lemma to establish inequalities involving $m$ and $n$. The resulting inequalities fail to hold for $n > 13$ in the Tchebycheff-Hermite case and for $n > 9$ in the Tchebycheff-Laguerre case, proving the impossibility of quadrature formulae of this type for large $n$. Combining these results with those of Salzer and Burgoyne, Gatteschi concludes that Tchebycheff quadrature is only possible in the Tchebycheff-Hermite case for $n=1,2,3$ and in the Tchebycheff-Laguerre case for $n=1,2$.

1.10 <u>METHOD OF H.S. WILF</u>

Wilf [25] has considered a more general type of quadrature formula

$$\int_a^b f(x)\, d\psi(x) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) \ , \int_a^b d\psi(x) = 1 \ .$$

Following Wilf, we say that a measure $d\psi$ has property T if Tchebycheff quadrature is possible on a sequence $(n_j)$, $(j=1,2,\ldots)$, of integers tending to infinity, and $\{n_j\}_{j=1}^{\infty}$ is called a T-sequence.

The following is a table of some of the more common measures and the corresponding T-sequences:

| $d\psi(x)$ | Interval (a,b) | T-sequence | Property T |
|---|---|---|---|
| $\frac{dx}{\pi\sqrt{1-x^2}}$ | $(-1,1)$ | all integers | yes |
| $dx$ | $(-1,1)$ | 1,2,3,4,5,6, 7,9 | no |
| $e^{-x}\ dx$ | $(0,\infty)$ | 1,2 | no |
| $\sqrt{\pi}\ e^{-x^2}\ dx$ | $(-\infty,\infty)$ | 1,2,3 | no |
| $\frac{1}{\pi\sqrt{1-x^2}}\ \frac{1+2ax\ dx}{1+4a^2+4ax}$ | $(-1,1)$ | all integers | yes |

The last table entry is a result due to Ullman and is not a commonly used measure except for $a = 0$.

Wilf has also shown that a necessary condition for a measure to have property T can be determined by employing Jensen's Inequality, viz:

If $\xi_1, \xi_2,\ldots,\xi_n$ are non-negative, and if we define

$$\sigma_r = \left\{ \sum_{\nu=1}^{n} \xi_\nu^r \right\}^{\frac{1}{r}} \quad (r=1,2,\ldots)$$

then $s < t$ implies $\sigma_s \geq \sigma_t$ .

Applying this inequality to

$$\mu_r = \int_a^b x^r \, d\psi(x) = \frac{1}{N} \sum_{i=1}^{N} x_i^r (r=1,\ldots,N)$$

he proves the Theorem: Let $\{n_j\}_{j=1}^{\infty}$ be the T-sequence of a measure with moments $\mu_r (r=0,1,\ldots)$. Then for each fixed $j (j=1,2,\ldots)$ the sequence

$$\tau_r = (n_j \mu_{2r})^{\frac{1}{r}} \quad (r=1,2,\ldots, \left[\frac{n_j}{2}\right])$$

decreases. If $a \geq 0$, then actually

$$\sigma_r = (n_j \mu_r)^{\frac{1}{r}} \quad (r=1,2,\ldots,n_j)$$

decreases.

Applying this theorem to the Tchebycheff-Laguerre formula, it is shown that $\sigma_r$ increases for $n > 2$ and hence Tchebycheff-Laguerre quadrature is impossible for $n > 2$. By a similar analysis, Wilf shows that Tchebycheff-Hermite quadrature is impossible for $n \geq 4$, from which he concludes that the measures of Hermite and Laguerre do not have property T.

It is also shown that if a measure has property T, then large gaps exist between the elements of the T-sequence. In addition, Wilf conjectured that if a measure has property T, then it has zero mass outside of some finite interval.

## 1.11 FABER POLYNOMIALS AND ULLMAN'S CONTRIBUTION TO THE PROBLEM.

Ullman [23] has disproved Wilf's conjecture by showing the existence of a unit mass distribution which has property T, but whose mass set is not located on any finite interval.

He defines a simple mass distribution of degree $n$ to be a unit mass distribution with $n$ equal masses located at $n$ distinct points. The mass points can then be regarded as the zeros of the Faber polynomials related to the Laurent series given by

$$z \exp\left(-\sum_{k=1}^{\infty} \frac{m_k}{kz^k}\right) = z + a_o + \frac{a_1}{z} + \ldots$$

and as the nodes of a quadrature formula valid for all polynomials of degree $\leq n$.

<u>Definition</u>: Let $f(z)$ have the Laurent series expansion

$$z + a_o + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots , \quad \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} < \infty$$

For each $n \geq 1$, form

$$[f(z)]^n = F_n(z) + R_n(z)$$

where $F_n(z)$ is a polynomial of degree $n$ and $R_n(z)$ contains only negative powers of $z$. The polynomials $F_n(z)$ are the <u>Faber polynomials</u> associated with $f(z)$. Thus if

$f(z) = z + \sqrt{z^2-1}$

$$F_n(z) = (z + \sqrt{z^2-1})^n + (z - \sqrt{z^2-1})^n .$$

Ullman describes a scheme for distributing the unit mass distribution over the infinite interval in such a manner that the limiting distribution does not have its mass set located on any finite interval and that the Faber polynomials associated with the distribution have real zeros. The nodes of the quadrature formula are thus real and we have the existence of a mass distribution on the infinite interval for which Tchebycheff quadrature is possible.

Ullman [24] has also shown that Tchebycheff quadrature is possible on [-1,1] with weight function

$$(1.11.1) \qquad w(x) = \frac{1}{\pi\sqrt{1-x^2}} \frac{1+2ax}{1+4a^2+4ax} , \quad -\frac{1}{4} \le a \le \frac{1}{4} .$$

Setting

$$m_k = \int_{-1}^{1} x^k \, w(x) \, dx$$

Ullman obtains the nodes as the zeros of the Faber polynomials $F_n(z)$ associated with

$$f(z) = z \exp \left(- \sum_{k=1}^{\infty} \frac{m_k}{kz^k}\right) .$$

For the weight (1.11.1), we have explicitly

$$f(z) = \frac{z+\sqrt{z^2-1}}{2} + a$$

$$F_n(z) = \left(\frac{z+\sqrt{z^2-1}}{2} + a\right)^n + \left(\frac{z-\sqrt{z^2-1}}{2} + a\right)^n - a^n.$$

If $a = 0$, then $F_n(z)$ becomes the normalized Tchebycheff polynomial of degree $n$, and we obtain Hermite's quadrature formula whose nodes are the zeros of the Tchebycheff polynomial. For $0 \leq |a| \leq \frac{1}{4}$ Ullman shows that the zeros of $F_n(z)$ are all real and hence <u>Tchebycheff quadrature is possible for all positive integral values of $n$</u>.

## 1.12 MODIFIED TCHEBYCHEFF QUADRATURE FORMULAE

We are thus faced with the situation of having n-point Gauss-type quadrature formulae valid for all polynomials of degree $\leq 2n-1$ and Tchebycheff-type quadrature formulae, being possible for very limited $n$. If we consider modified formulae resulting from a combination of Gauss-type and Tchebycheff-type formulae, two questions arise naturally and as formulated by P. Erdös and A. Sharma [3], they are:

<u>Problem 1</u>: Given a fixed integer $k$, we wish to determine $n+1+k (n \geq k+2)$ constants $A_i$, $y_i (i=1,2,\ldots,k)$, $x_j (j=1,2,\ldots n-k)$, and $B$ so that the formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{k} A_i\, f(y_i) + B \sum_{j=1}^{n-k} f(x_j)$$

is exact for all polynomials of degree $\leq n+k$. We require the $y_i$'s and $x_j$'s to be in $[-1,1]$. Does there exist a number $n_o$ such that for $n > n_o$ the formula is no longer valid?

Problem 2: If for every $n$, the formula of Problem 1 is required to be valid for all polynomials of degree $m = m(n) < n$, what is the order of $m(n)$?

It is shown in [3] that if the conditions of Problem 1 are satisfied, the answer to Problem 2 is

$$m(n) \leq c_k \sqrt{n}$$

where $c_k$ depends only on $k$. Thus Problem 1 is answered in the affirmative since the quadrature formula cannot be valid in general for polynomials of degree $n + k$ if $n$ is sufficiently large.

A. Meir and A. Sharma [13] considered the validity (for polynomials) of the more general formula

$$\int_{-1}^{1} w(x)\, f(x)\,dx = \sum_{i=1}^{k} A_i\, f(y_i) + B \sum_{i=1}^{n-k} f(x_i) \tag{1.12.1}$$

where $w(x)$ is a non-negative weight function. Their results are contained in the following:

<u>Theorem</u>. Let $k$ be a fixed non-negative integer, $w(x) = (1-x^2)^\alpha$, $\alpha > -1$. Then for the degree of exactness $N = N(n,k, )$ of a formula of type (1.12.1), we have

$$N < Cn^{\frac{1}{2\alpha+2}}$$

where $C = C(k,\alpha)$ is independent of $n$.

Similarily if $w(x) = (1-x)^a(1+x)^b$, $a,b$ are real and $> -1$, Jimmie Jones [7] has shown that if in (1.12.1), the $A_i$ are assumed to be non-negative, then for the degree of exactness $N$ we have

$$N \leq C(k,a,b)\, n^{\frac{1}{2\{\max\,(a,b)\}+2}}$$

where $C(k,a,b)$ is a constant independent of $n$. J. Jones also shows that there exist two distinct quadrature formulae of the type

$$\int_{-1}^{1} f(x)\, dx = Af(y) + B \sum_{i=1}^{3} f(x_i)$$

which are exact for all polynomials of degree $< 5$.

# CHAPTER 2

# MODIFIED TCHEBYCHEFF QUADRATURE ON $(-\infty,\infty)$

## 2.1 INTRODUCTION

In Chapter 1, we have discussed the problem of modified Tchebycheff quadrature on $[-1,1]$ with weight function $w(x) = 1$ (Erdös and Sharma [3]) and weight function $w(x) = (1 - x^2)^\alpha$, $\alpha > -1$ (Meir and Sharma [13]). However, the corresponding problem for the infinite interval has not been considered. In the present Chapter we consider the modified Tchebycheff-Hermite quadrature formula:

$$(2.1.1) \qquad \int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} f(x)\, dx = Af(y) + B \sum_{i=1}^{n-1} f(x_i)$$

and consider, for this formula, problems of the type posed in Section 1.12 for the finite interval. Our results can be formulated by means of:

Theorem 1. If a formula of type (2.1.1) is valid for all polynomials of degree $\leq 2m+1$, $m < n$, with real $A$, $B$, $y$ and $x_i (i=1,2,\ldots,n-1)$, and $\alpha > -\frac{1}{2}$, then

$$m < C \log n$$

where $C = C(\alpha)$ is independent of $n$.

The case when $\alpha = 0$ is of some interest in itself, being connected with Tchebycheff quadrature using the common Hermite weight function on the infinite interval. We then state:

Corollary 1. *Assuming the formula*

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = Af(y) + B \sum_{i=1}^{n-1} f(x_i)$$

*to be valid for all polynomials of degree* $\leq 2m+1$, $m < n$, *with real* $A$, $B$, $y$, *and* $x_i (i=1,2,\ldots,n-1)$, *then*

$$m < C \log n$$

*where* $C$ *is a constant independent of* $n$.

## 2.2 PRELIMINARIES

In order to prove Theorem 1, we shall introduce the polynomials $H_n^{(\alpha)}(x)$ which are a natural generalization of Hermite polynomials (Szegö [19] p. 377). $H_n^{(\alpha)}(x)$ are orthogonal polynomials corresponding to the weight function $e^{-x^2}|x|^{2\alpha}$, $\alpha > -\frac{1}{2}$ in $[-\infty,\infty]$. Their relation to the Laguerre polynomials is given by

$$(2.2.1)\quad \begin{cases} H_{2m}^{(\alpha)}(x) = (-1)^m\, 2^{2m}\, m!\ L_m^{(\alpha-\frac{1}{2})}(x^2), & \alpha > -\frac{1}{2} \\[2ex] H_{2m+1}^{(\alpha)}(x) = (-1)^m\, 2^{2m+1}\, m!\ xL_m^{(\alpha+\frac{1}{2})}(x^2), & \alpha > -\frac{1}{2}\ . \end{cases}$$

The case when $a = 0$ is of some interest in itself, [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] of quadrature.

Conjecture 1. [illegible] the formula

$$[illegible]$$

to be valid for all polynomials of degree [illegible] [illegible] [illegible] [illegible]

$$m \leq [illegible] n$$

Part 3. [illegible]

3.1 Preliminaries

In order to prove Theorem 1, we shall introduce the polynomials $[illegible]$ which are a [illegible] generalization of [illegible] polynomials [illegible] [illegible] [illegible] [illegible] orthogonal polynomials corresponding to the [illegible] function [illegible] whose generating function is given by

$$[illegible] \qquad (3.1.1)$$

Denoting the zeros of $H_m^{(\alpha)}(x)$ by

$$\xi_{1,m}^{(\alpha)} < \xi_{2,m}^{(\alpha)} < \cdots < \xi_{m,m}^{(\alpha)}$$

and the zeros of $L_m^{(\alpha)}(x)$ by

$$\mu_{1,m}^{(\alpha)} < \mu_{2,m}^{(\alpha)} < \cdots < \mu_{m,m}^{(\alpha)}$$

we have from (2.2.1) and Szegö's estimate([19] p. 118) on the largest zero of the Laguerre polynomial $L_n^{(\alpha)}(x)$ ,

$$\text{(2.2.2)} \qquad \begin{cases} [\xi_{2m,2m}^{(\alpha)}]^2 = \mu_{m,m}^{(\alpha-\frac{1}{2})} > (2m+\alpha-3/2) \\ \\ [\xi_{2m+1,2m+1}^{(\alpha)}]^2 = \mu_{m,m}^{(\alpha+\frac{1}{2})} > (2m+\alpha-\frac{1}{2}) \ . \end{cases}$$

Combining the estimates of (2.2.2) we obtain

$$\text{(2.2.3)} \qquad \xi_{1,m}^{(\alpha)} < -(m+\alpha-3/2)^{\frac{1}{2}} \ .$$

We shall also require Stirling's formula for the Gamma function:

$$\text{(2.2.4)} \qquad \Gamma(m+1) < 2\sqrt{\pi m}\,(\tfrac{m}{e})^m \ .$$

## 2.3 SOME LEMMAS

We shall require the following lemmas in the proof of Theorem 1.

Lemma 1. *If* (2.1.1) *is valid for polynomials of degree* $\leq N$, *then* $B > 0$.

Proof. Examining (2.1.1) we see that $N > 2$ and choosing $f(x) = (x-y)^2$, we have

$$0 < \int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} (x-y)^2 \, dx = B \sum_{i=1}^{n-1} (x_i - y)^2 .$$

Since $(x_i - y)^2 > 0$, we must have $B > 0$.

Lemma 2. *If formula* (2.2.1) *is valid with* $A, B, y$ *and* $x_i (i=1,2,\ldots,n-1)$ *for polynomials of degree* $\leq N$, *then it is valid for polynomials of degree* $\leq N$ *with* $A, B, -y$ *and* $-x_i (i=1,2,\ldots,n-1)$.

Proof. Choosing $f(x) = x^\nu$, $0 \leq \nu \leq N$, we have

$$\int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} x^\nu \, dx = Ay^\nu + B \sum_{i=1}^{n-1} x_i^\nu . \tag{2.3.1}$$

For $\nu$ an even integer, the Lemma is obvious. If $\nu$ is an odd integer we have $\int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} x^\nu \, dx = 0$ so we can replace $y$ by $-y$ and $x_i (i=1,2,\ldots,n-1)$ by $-x_i (i=1,2,\ldots,n-1)$. Lemma 2 is thus proven.

Lemma 3. Let $H_m^{(\alpha)}(x)$, $\alpha > -\frac{1}{2}$, denote the orthogonal polynomials of §2.2 and let $\{\xi_{i,m}^{(\alpha)}\}_{i=1}^{m}$ be the zeros of $H_m^{(\alpha)}(x)$.

If $y \geq 0$ and $R(x)$ is the polynomial of degree $2m+1$ given by

$$(2.3.2)\qquad R(x) = \frac{[H_m^{(\alpha)}(x)]^2(x-y)^2}{(x-\xi_{1,m}^{(\alpha)})}, \quad m \geq 2$$

then

$$(2.3.3)\qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} R(x)\,dx < 0 .$$

Proof. $R(x)$ may be written as

$$(2.3.4)\quad R(x) = [H_m^{(\alpha)}(x)]^2(x+\xi_{1,m}^{(\alpha)}-2y)+(\xi_{1,m}^{(\alpha)}-y)^2 \frac{[H_m^{(\alpha)}(x)]^2}{(x-\xi_{1,m}^{(\alpha)})} .$$

Since $[H_m^{(\alpha)}(x)]^2$ and $e^{-x^2}|x|^{2\alpha}$ are both even, it follows that

$$(2.3.5)\qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} x[H_m^{(\alpha)}(x)]^2\,dx = 0 .$$

By the orthogonality of the $H_m^{(\alpha)}(x)$,

$$(2.3.6)\qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} \frac{[H_m^{(\alpha)}(x)]^2}{(x-\xi_{1,m}^{(\alpha)})}\,dx = 0 .$$

Multiplying (2.3.4) by $e^{-x^2}|x|^{2\alpha}$ and integrating, we obtain

$$(2.3.7)\quad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} R(x)\,dx = (\xi_{1,m}^{(\alpha)}-2y)\int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha}[H_m^{(\alpha)}(x)]^2 dx\ .$$

For $m \geq 2$, $\xi_{1,m}^{(\alpha)} < 0$, whence the result (2.2.4) follows.

Lemma 4. If formula (2.1.1) is exact for all polynomials of degree $\leq 2m+1$, $m < n$, and if the abscissae are given by $x_1 < x_2 < \cdots < x_{n-1}$, then we have

$$x_1 < \xi_{1,m}^{(\alpha)}\ .$$

Proof. By Lemma 2, we may assume without loss of generality, that $y \geq 0$. Then for the polynomial $R(x)$ defined by (2.3.2) we have from (2.3.3)

$$(2.3.8)\qquad 0 > \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} R(x)\,dx = B\sum_{i=1}^{n-1} R(x_i)\ .$$

Therefore, since $B > 0$ by Lemma 1, there exists a $\nu$ such that $R(x_\nu) < 0$. From (2.3.2), $R(x) < 0$ only if $x < \xi_{1,m}^{(\alpha)}$. Hence $x_\nu < \xi_{1,m}^{(\alpha)}$ for some $\nu$ and $x_1 < \xi_{1,m}^{(\alpha)}$.

## 2.4 PROOF OF THEOREM 1

Proceeding with the proof of Theorem 1, we can choose $y > 0$ without loss of generality by Lemma 2. We will

Substituting (3.2.11) in [illegible] and simplifying, we obtain

(3.2.13) [illegible]

For $n \geq k$, $\binom{n}{k}$ [illegible]

**Lemma 3.** If [illegible]

[illegible]

$$\binom{2n}{n}$$

*Proof.* [illegible]

that $k \leq 2$. [illegible]

(3.2.5) we have from [illegible]

(3.2.14) [illegible]

[illegible]

## 3.3 [illegible]

[illegible]

consider the proof in two cases determined by the location of $y$.

<u>Case 1</u>. Suppose $|x_1-y| \geq |x_{n-1}-y|$.

Choosing $f(x) = (\frac{x-y}{x_1-y})^2$ and using (2.1.1) we have

$$\text{(2.4.1)} \qquad nB \geq B \sum_{i=1}^{n-1} (\frac{x_i-y}{x_1-y})^2 = \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha} (\frac{x-y}{x_1-y})^2 \, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x^2}|x|^{2\alpha}(x^2+y^2)}{(x_1-y)^2} \, dx$$

$$= \int_{0}^{\infty} \frac{e^{-t}t^{\alpha-\frac{1}{2}}(t+y^2)\, dt}{(x_1-y)^2}$$

$$= \frac{\Gamma(\alpha+3/2) + y^2\,\Gamma(\alpha+\frac{1}{2})}{(x_1-y)^2}$$

$$> \frac{C_1(1+y^2)}{(x_1-y)^2} .$$

From (2.4.1), we obtain a lower bound on $B$:

$$\text{(2.4.2)} \qquad B > \frac{C_1(1+y^2)}{n(x_1-y)^2} .$$

We now choose $f(x) = (\frac{x-y}{x_1-y})^2(\frac{x}{x_1})^{2m-6}$ and again use formula (2.1.1) to obtain

$$(2.4.3) \quad \int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} \left(\frac{x-y}{x_1-y}\right)^2 \left(\frac{x}{x_1}\right)^{2m-6} dx = B \sum_{i=1}^{n-1} \left(\frac{x_i-y}{x_1-y}\right)^2 \left(\frac{x_i}{x_1}\right)^{2m-6}$$

$$\geq B \; .$$

Proceeding to evaluate the left hand side of (2.4.3) we obtain the following upper bound for $B$ in terms of $m$:

$$\int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} \left(\frac{x-y}{x_1-y}\right)^2 \left(\frac{x}{x_1}\right)^{2m-6} dx = \int_{-\infty}^{\infty} \frac{e^{-x^2} |x|^{2\alpha} (x^{2m-4} + y^2 x^{2m-6})}{(x_1-y)^2 \; x_1^{2m-6}} dx$$

$$(2.4.4) \qquad < \frac{\Gamma(m+\alpha-3/2) + y^2 \; \Gamma(m+\alpha-5/2)}{(x_1-y)^2 \; x_1^{2m-6}}$$

$$< \frac{C_2(1+y^2) \; \Gamma(m+\alpha-3/2)}{(x_1-y)^2 \; x_1^{2m-6}} \; .$$

Employing (2.2.3) and Lemma 4, we have

$$(2.4.5) \qquad x_1 < -(m+\alpha-3/2)^{\frac{1}{2}} \; .$$

Using Stirling's formula for the Gamma function and (2.4.5), we can express (2.4.4) as

$$\int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha}\left(\frac{x-y}{x_1-y}\right)^2\left(\frac{x}{x_1}\right)^{2m-6} dx < \frac{C_3(1+y^2)(m+\alpha-5/2)^{m+\alpha-2}}{e^{m+\alpha-5/2}(m+\alpha-3/2)^{m-3}(x_1-y)^2}$$

(2.4.6)

$$< \frac{C_4(1+y^2)(m+\alpha-5/2)^{\alpha+1}}{e^m(x_1-y)^2} .$$

Combining (2.4.2), (2.4.3) and (2.4.6) we have the following inequality involving B:

(2.4.7) $$\frac{C_4(1+y^2)(m+\alpha-5/2)^{\alpha+1}}{e^m(x_1-y)^2} > B \geq \frac{C_1(1+y^2)}{n(x_1-y)^2} .$$

Simplifying (2.4.7) we have

(2.4.8) $$\frac{C_5(m+\alpha-5/2)^{\alpha+1}}{e^m} > \frac{1}{n} .$$

Taking logarithms of (2.4.8) yields

$$C_6 + (\alpha+1) \cdot \log (m+\alpha-5/2) - m > - \log n$$

which can be simplified to

(2.4.9) $$m < C_7 \log n$$

where $C_7 = C_7(\alpha)$ .

<u>Case 2</u>. Suppose $|x_{n-1}-y| > |x_1-y|$ .

Since $y > 0$, we must have

$$|x_{n-1}| > |x_1| > |\xi_{1,m}^{(\alpha)}| .$$

Proceeding as in Case 1, we choose $f(x) = (\frac{x-y}{x_{n-1}-y})^2$ and arrive at

$$B > \frac{C_8(1+y^2)}{n(x_{n-1}-y)^2} . \tag{2.4.10}$$

If we then choose

$$f(x) = (\frac{x-y}{x_{n-1}-y})^2(\frac{x}{x_{n-1}})^{2m-6}$$

we obtain, as in Case 1, the following inequality:

$$\frac{C_9(1+y^2)(m+\alpha-5/2)^{\alpha+1}}{e^m(x_{n-1}-y)^2} > B . \tag{2.4.11}$$

Combining (2.4.10) and (2.4.11) and simplifying, we have

$$\frac{C_{10}(m+\alpha-5/2)^{\alpha+1}}{e^m} > \frac{1}{n} . \tag{2.4.12}$$

Taking logarithms of (2.4.12), we obtain

$$m < C_{11} \log n \tag{2.4.13}$$

where $C_{11} = C_{11}(\alpha)$ .

Choosing $C$ to be the larger of $C_7$ in (2.4.9) and $C_{11}$ in (2.4.13) completes the proof of Theorem 1.

# CHAPTER 3

# QUADRATURE FORMULAE FOR THE SEMI-INFINITE AND INFINITE INTERVAL

## 3.1 INTRODUCTION

In this chapter, we will consider the infinite Tchebycheff quadrature formula

$$(3.1.1) \qquad \int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} f(x)\, dx = B \sum_{i=1}^{n} f(x_i)$$

and the semi-infinite Tchebycheff quadrature formula

$$(3.1.2) \qquad \int_{0}^{\infty} e^{-x} x^{\alpha} f(x)\, dx = B \sum_{i=1}^{n} f(x_i) .$$

Wilf [25] and Gatteschi [5] have considered the case $\alpha = 0$, and their results are summarized in Chapter 1. However, they have not considered the degree of exactness $N$ of these formulae. Our estimates for $N$ are contained in Theorems 1 and 2, which give a qualitative generalization of the results of Gatteschi and Wilf for the infinite and semi-infinite intervals.

## 3.2 INFINITE INTERVAL

<u>Theorem 1</u>. <u>If a formula of type</u> (3.1.1) <u>is valid for polynomials of degree</u> $\leq 2m-1$, $m < n$, <u>with real</u> $B$ <u>and real</u> $x_i (i=1,2,\ldots,n)$, <u>and</u> $\alpha > -\frac{1}{2}$ <u>then</u>

$$m < K \log n$$

<u>for some constant</u> $K = K(\alpha)$ .

We will require the following lemma for the proof of Theorem 1 and will use the notation of §2.2.

<u>Lemma 1.</u> <u>If formula</u> (3.1.1) <u>is valid for polynomials of degree</u> $\leq 2m-1$, <u>and if the</u> $x_i (i=1,2,...,n)$ <u>satisfy</u>

$$x_1 < x_2 < \cdots < x_n$$

<u>then</u>

$$x_1 < \xi_{1,m}^{(\alpha)} \quad .$$

<u>Proof</u>. Choosing $f(x) = \dfrac{[H_m^{(\alpha)}(x)]^2}{x-\xi_{1,m}^{(\alpha)}}$ , a polynomial of degree 2m-1, we can employ the orthogonal property of the $H_m^{(\alpha)}(x)$ and formula (3.1.1) to obtain

$$(3.2.1) \quad 0 = \int_{-\infty}^{\infty} e^{-x^2} |x|^{2\alpha} \frac{[H_m^{(\alpha)}(x)]^2}{x-\xi_{1,m}^{(\alpha)}} dx = \frac{\Gamma(\alpha+\frac{1}{2})}{n} \sum_{i=1}^{n} \frac{[H_m^{(\alpha)}(x_i)]^2}{x_i-\xi_{1,m}^{(\alpha)}} \quad .$$

Since $[H_m^{(\alpha)}(x_i)]^2 \geq 0$ and $H_m^{(\alpha)}(x_i) \not\equiv 0$, for all $i$ , $(x_i-\xi_{1,m}^{(\alpha)}) < 0$ for some $x_i$ . Hence, $(x_1-\xi_{1,m}^{(\alpha)}) < 0$ which yields the desired inequality

$$x_1 < \xi_{1,m}^{(\alpha)} \quad .$$

Proof of Theorem 1.

Returning to the proof of Theorem 1, we will now establish an upper bound on the constant $\frac{\Gamma(\alpha+\frac{1}{2})}{n}$ in terms of $m$.

Since we have assumed (3.1.1) to be exact for polynomials of degree $\leq 2m-1$, we may choose $f(x) = (\frac{x}{x_1})^{2m-2}$. Then from (3.1.1), we have

$$(3.2.2) \qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha}(\frac{x}{x_1})^{2m-2}\,dx = \frac{\Gamma(\alpha+\frac{1}{2})}{n}\sum_{i=1}^{n}(\frac{x_i}{x_1})^{2m-2}$$

$$> \frac{\Gamma(\alpha+\frac{1}{2})}{n}.$$

Evaluating the left hand side of (3.2.2) we have

$$(3.2.3) \qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha}(\frac{x}{x_1})^{2m-2}\,dx = \int_{0}^{\infty}\frac{e^{-t}t^{m+\alpha-3/2}}{x_1^{2m-2}}\,dt$$

$$= \frac{\Gamma(m+\alpha-\frac{1}{2})}{x_1^{2m-2}}.$$

Employing Stirling's formula in (3.2.3) we obtain

$$(3.2.4) \qquad \int_{-\infty}^{\infty} e^{-x^2}|x|^{2\alpha}(\frac{x}{x_1})^{2m-2}\,dx < \frac{C_1(m+\alpha-3/2)^{m+\alpha-1}}{e^{m+\alpha-3/2}\,x_1^{2m-2}},$$

Combining the estimates of (3.2.2) and (3.2.4) yields

$$(3.2.5) \qquad \frac{C_1(m+\alpha-3/2)^{m+\alpha-1}}{e^{m+\alpha-3/2}\; x_1^{2m-2}} > \frac{\Gamma(\alpha+\frac{1}{2})}{n} .$$

Employing the estimate (2.2.3) and Lemma 1, we have

$$(3.2.6) \qquad x_1 < -(m+\alpha-3/2)^{\frac{1}{2}}.$$

Substituting the above inequality in (3.2.5) yields

$$(3.2.7) \qquad \frac{\Gamma(\alpha+\frac{1}{2})}{n} < \frac{C_1(m+\alpha-3/2)^{m+\alpha-1}}{e^{m+\alpha-3/2}(m+\alpha-3/2)^{m-1}}$$

$$< \frac{C_2(m+\alpha-3/2)^{\alpha}}{e^m} .$$

Taking logarithms of (3.2.7) yields

$$C_4 - \log n < C_5 + (\alpha) \log (m+\alpha-3/2) - m$$

which can be written as

$$(3.2.8) \qquad m < K \log n$$

where $K = K(\alpha)$, completing the proof of Theorem 1.

### 3.3 SEMI-INFINITE INTERVAL

Theorem 2. If formula (3.1.2) is valid for polynomials of degree $\leq 2m-1$, $m < n$, with real $B$ and with real positive $x_i (i=1,2,\ldots,n), \alpha > -1$ then

$$m < K \log n$$

<u>for some constant</u> $K = K(\alpha)$.

We shall require the following lemma in the proof of Theorem 2 and will use the notation of §2.2.

<u>Lemma 1</u>. <u>If formula</u> (3.1.2) <u>is valid for all polynomials of degree</u> $\leq 2m-1$, $m < n$, <u>and if the nodes</u> $x_i (i=1,2,\ldots,n)$ <u>satisfy</u>

$$x_1 < x_2 < \cdots < x_n$$

<u>and</u> $\alpha > -1$, <u>then</u>

$$x_n > \mu_{m,m}^{(\alpha)} . \tag{3.3.1}$$

<u>Proof</u>. Choosing $f(x) = \dfrac{[L_m^{(\alpha)}(x)]^2}{(x-\mu_{m,m}^{(\alpha)})}$, a polynomial of degree $2m-1$, we can employ the orthogonal property of the $L_m^{(\alpha)}(x)$ and formula (3.1.2) to obtain

$$0 = \int_0^\infty e^{-x} x^\alpha f(x)\, dx = \frac{\Gamma(\alpha+1)}{n} \sum_{i=1}^{n} \frac{[L_m^{(\alpha)}(x_i)]^2}{(x_i-\mu_{m,m}^{(\alpha)})} .$$

Since $[L_m^{(\alpha)}(x_i)]^2 \geq 0$ and $[L_m^{(\alpha)}(x_i)] \not\equiv 0$, for all $i$, $(x_i - \mu_{m,m}^{(\alpha)}) > 0$ for some $x_i$.

Hence $(x_n-\mu_{m,m}^{(\alpha)}) > 0$ which yields the desired inequality

$$x_n > \mu_{m,m}^{(\alpha)} .$$

[illegible]

The upper estimate. [illegible]

We shall require the following lemma in the proofs of Theorems 8 and 9(i) for the estimation of [illegible].

Lemma 13. If [illegible] is a polynomial with nonnegative coefficients [illegible], and [illegible]

[illegible]

[illegible]

[illegible]

Proof. [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Proof of Theorem 2.

We now proceed with the proof of Theorem 2 using the method of Theorem 1.

Since (3.1.2) is assumed to be exact for polynomials of degree $\leq 2m-1$, we can choose $f(x) = (\frac{x}{x_n})^{2m-1}$ and obtain from (3.1.2)

$$(3.3.2) \qquad \int_0^\infty e^{-x}x^\alpha(\frac{x}{x_n})^{2m-1}\,dx = \frac{\Gamma(\alpha+1)}{n}\sum_{i=1}^{n}(\frac{x_i}{x_n})^{2m-1} > \frac{\Gamma(\alpha+1)}{n}.$$

We also have

$$(3.3.3) \qquad \int_0^\infty e^{-x}x^\alpha(\frac{x}{x_n})^{2m-1}\,dx = \frac{\Gamma(2m+\alpha)}{x_n^{2m-1}}.$$

Using Stirling's formula in (3.3.3), the resulting inequality is

$$(3.3.4) \qquad \int_0^\infty e^{-x}x^\alpha(\frac{x}{x_n})^{2m-1}\,dx < \frac{C_1(2m+\alpha-1)^{2m+\alpha-\frac{1}{2}}}{e^{2m+\alpha-1}\,x_n^{2m-1}}.$$

Combining Lemma 1 and (2.2.2) we have

$$x_n > 2m+\alpha-1.$$

Substituting this inequality in (3.3.4) yields

$$(3.3.5)\qquad \int_0^\infty e^{-x} x^\alpha \left(\frac{x}{x_n}\right)^{2m-1} dx < \frac{C_2(2m+\alpha-1)^{2m+\alpha-\frac{1}{2}}}{e^{2m+\alpha-1}(2m+\alpha-1)^{2m-1}}$$

$$< \frac{C_3(2m+\alpha-1)^{\alpha+\frac{1}{2}}}{e^{2m}} .$$

Combining (3.3.2) and (3.3.5) we have

$$(3.3.6)\qquad \frac{C_4(2m+\alpha-1)^{\alpha+\frac{1}{2}}}{e^{2m}} > \frac{1}{n} .$$

Taking logarithms of (3.3.6) gives the inequality

$$(3.3.7)\qquad m < K \log n$$

where $K = K(\alpha)$ which completes the proof of Theorem 2.

CHAPTER 4

# PARTIAL RESULT AND OPEN PROBLEM

## 4.1 INTRODUCTION

In this chapter, we consider the modified Tchebycheff quadrature formula on the semi-infinite interval $(0,\infty)$:

$$(4.1.1) \qquad \int_0^\infty e^{-x} f(x)\, dx = A\, f(y) + B \sum_{i=1}^{n-1} f(x_i)$$

and propose the same problem as for the corresponding formula (2.1.1) for the interval $(-\infty,\infty)$ in Chapter 2. Our results in this connection are partial and incomplete. Although we believe that the analogue of Theorem 1, Chapter 2 holds true for formula (4.1.1), we have not been able to prove this so far. Hence the analogue of Theorem 1, Chapter 2 is formulated here as a conjecture in Theorem 2.

One of the reasons for the failure of an extension of our method of proof of Theorem 1 of Chapter 2 is that we can no longer determine the location of $y$ in relation to the larger nodes of the above quadrature formula. In other words we cannot establish that $\left(\frac{x_1-y}{x_N-y}\right) < K$ where $K$ is a constant independent of $n$, for some $x_N$ sufficiently large, and hence we can not establish inequalities of the type (2.4.4).

We shall now prove the following:

Theorem 1. *If* (4.1.1) *is exact for all polynomials of degree* $\leq 2m+1$, $m < n$, *with* $A$ *and* $B$ *real*, $y$ *and* $x_i (i=1,2,\ldots,n-1)$ *contained in* $(0,\infty)$, *and* $\left(\frac{x_1-y}{x_{n-1}-y}\right)^2 < C$ *where* $C>1$ *is a constant independent of* $n$, *then*

$$m < K \log n$$

*for some constant* $K$ *independent of* $n$.

## 4.2 SOME LEMMAS

We shall require the following lemmas in the proof of Theorem 1.

Lemma 1. *If* (4.1.1) *is valid for polynomials of degree* $\leq N$, *then* $B > 0$.

Proof. Examining (4.1.1), we see that $N \geq 2$ and choosing $f(x) = (x-y)^2$ gives the result.

Lemma 2. *The degree of exactness* $N$ *of formula* (4.1.1) *is a non-decreasing function of* $n$.

Proof. Suppose formula (4.1.1) is valid for polynomials of degree $\leq N$. Then defining

$$A^{(n+1)} = A^{(n)} - B^{(n)} \quad , \quad B^{(n+1)} = B^{(n)}$$

$$x_i^{(n+1)} = x_i^{(n)} \quad (1 \leq i \leq n-1)$$

$$x_n^{(n+1)} = y^{(n)} \quad , \quad y^{(n+1)} = y^{(n)}$$

we see that a formula of type (4.1.1) with $n$ replaced by $(n+1)$ is valid for all polynomials of degree $\leq N$ and hence the degree of exactness is a non-decreasing function of $n$.

Let the zeros of the Laguerre polynomial $L_m(x)$ be denoted by

$$\mu_{1,m} < \mu_{2,m} < \cdots < \mu_{m,m}$$

and let $L_m(x)$ be normalized such that

$$\int_0^\infty e^{-x} \; L_m^2(x) \, dx = 1 .$$

<u>Lemma 3</u>. <u>If</u> (4.1.1) <u>is exact for polynomials of degree</u> $\leq 2m+1$, <u>and</u> $y < \frac{\mu_{m,m}}{2}$ <u>then</u>

$$x_{n-1} > \mu_{m,m} .$$

<u>Proof</u>. Consider the function $R(x) = \dfrac{L_m^2(x)(x-y)^2}{(x-\mu_{m,m})}$.

From (4.1.1) we have

$$\int_0^\infty e^{-x} R(x) \, dx = B \sum_{i=1}^{n-1} R(x_i) . \tag{4.2.1}$$

Since we can write $R(x)$ as

$$R(x) = L_m^2(x)[x-y+(\mu_{m,m}-y)(1+\frac{\mu_{m,m}}{x-\mu_{m,m}})-y(\frac{\mu_{m,m}-y}{x-\mu_{m,m}})]$$

we have

$$\int_0^\infty e^{-x} R(x)\, dx = \int_0^\infty e^{-x} L_m^2(x)[x-y+(\mu_{m,m}-y)(1+\frac{\mu_{m,m}}{x-\mu_{m,m}})-y(\frac{\mu_{m,m}-y}{x-\mu_{m,m}})]$$

$$= \int_0^\infty e^{-x} L_m^2(x)\, xdx - 2y \int_0^\infty e^{-x} L_m^2(x)\, dx$$

(4.2.2)

$$+ \mu_{m,m} \int_0^\infty e^{-x} L_m^2(x)\, dx$$

$$> - 2y + \mu_{m,m}$$

$$> 0 .$$

Hence we have from (4.2.1)

$$B \sum_{i=1}^{n-1} R(x_i) > 0 . \tag{4.2.3}$$

Since $B > 0$ by Lemma 1, (4.2.3) will be true only if

$$\sum_{i=1}^{n-1} \frac{L_m^2(x_i)(x_i-y)^2}{(x_i-\mu_{m,m})} > 0 \tag{4.2.4}$$

and (4.2.4) will hold only if $x_\nu > \mu_{m,m}$ for some $\nu$ and hence

$$x_{n-1} > \mu_{m,m} .$$

Lemma 4. *If* (4.1.1) *is exact for all polynomials of degree* $\le 2m+1$, $5 \le m < n$, *then*

$$x_{n-1} > \frac{\mu_{m,m}}{4} .$$

Proof. If $y < \frac{\mu_{m,m}}{2}$ we can use Lemma 2 to obtain the desired inequality.

We will now consider the proof for $y \ge \frac{\mu_{m,m}}{2}$. Let us assume that $x_{n-1} \le \frac{\mu_{m,m}}{4}$. We have (Szegö [19] pp. 127) $\mu_{m,m} < 4m+3$ and (Szegö [19] pp. 126)

$$\mu_{m-1,m} > \frac{\pi^2}{4} \frac{(m-5/4)^2}{m+1/2} .$$

Since

$$\frac{4m+3}{4} < \frac{\pi^2}{4} \frac{(m-5/4)^2}{m+1/2} \qquad m \ge 5$$

we have

$$\frac{\mu_{m,m}}{4} < \mu_{m-1,m} .$$

Choosing $f(x) = \frac{L_m^2(x)(x-y)}{(x-\mu_{m,m})(x-\mu_{m-1,m})}$, a polynomial of degree $2m-1$, we have from (4.1.1)

(4.2.5)

$$0 = \int_0^\infty \frac{e^{-x}L_m^2(x)(x-y)}{(x-\mu_{m,m})(x-\mu_{m-1,m})}\,dx = B \sum_{i=1}^{n-1} \frac{L_m^2(x_i)(x_i-y)}{(x_i-\mu_{m,m})(x_i-\mu_{m-1,m})} .$$

The right hand side contains only negative terms which is impossible. Hence we have a contradiction and

$$x_{n-1} > \frac{\mu_{m,m}}{4} .$$

## 4.3 PROOF OF THEOREM 1

By Lemma 2, it is sufficient to prove the theorem when $m \geq 5$ and we choose $n$ in (4.1.1) to be a sufficiently large integer. Choosing $f(x) = (\frac{x-y}{x_{n-1}-y})^2$ we have from (4.1.1)

$$(4.3.1) \qquad n\, C_o\, B \geq B \sum_{i=1}^{n-1} (\frac{x_i-y}{x_{n-1}-y})^2 = \int_0^\infty e^{-x}(\frac{x-y}{x_{n-1}-y})^2\, dx$$

$$= \frac{C_1(1-y+y^2)}{(x_{n-1}-y)^2}$$

and hence

$$(4.3.2) \qquad B \geq \frac{C_2(1-y+y^2)}{n(x_{n-1}-y)^2} .$$

Choosing $f(x) = (\frac{x-y}{x_{n-1}-y})^2(\frac{x}{x_{n-1}})^{m-3}$ and substituting in (4.1.1) we have

$$(4.3.3) \quad \int_0^\infty e^{-x}(\frac{x-y}{x_{n-1}-y})^2(\frac{x}{x_{n-1}})^{m-3}\, dx = B \sum_{i=1}^{n-1} (\frac{x_i-y}{x_{n-1}-y})^2(\frac{x_i}{x_{n-1}})^{m-3} \geq B .$$

We now obtain an upper bound for $B$ by finding an upper bound for the left hand side of (4.3.3) as follows:

$$\int_0^\infty e^{-x}\left(\frac{x-y}{x_{n-1}-y}\right)^2\left(\frac{x}{x_{n-1}}\right)^{m-3} dx = \int_0^\infty \frac{e^{-x}(x^{m-1}-yx^{m-2}+y^2x^{m-3})}{(x_{n-1}-y)^2(x_{n-1})^{m-3}}\, dx$$

(4.3.4)
$$< \frac{C_3\,\Gamma(m)(1-y+y^2)}{(x_{n-1}-y)^2(x_{n-1})^{m-3}}.$$

Combining Lemma 4 and the bound on $\mu_{m,m}$ given by (2.2.2) we have

(4.3.5)
$$x_{n-1} > \frac{2m-1}{4}.$$

Using Stirling's formula for the Gamma function and (4.3.5) in (4.3.4) we have

(4.3.6)
$$\int_0^\infty e^{-x}\left(\frac{x-y}{x_{n-1}-y}\right)^2\left(\frac{x}{x_{n-1}}\right)^{m-3} dx < \frac{C_4(m-1)^{m-\frac{1}{2}}(1-y+y^2)}{(x_{n-1}-y)^2\; e^{m-1}\left(\frac{(m-\frac{1}{2})}{2}\right)^{m-3}}.$$

Combining (4.3.2), (4.3.3), and (4.3.6) we have

$$\frac{C_5(m-1)^{5/2}(1-y+y^2)2^{m-3}}{(x_{n-1}-y)^2\; e^m} > B \geq \frac{C_2(1-y+y^2)}{n(x_{n-1}-y)^2}$$

which can be simplified to

(4.3.7)
$$\frac{C_6\; m^{5/2}\; 2^{m-3}}{e^m} > \frac{1}{n}.$$

Taking logarithms of (4.3.7) yields

$$C_7 + 5/2 \log m + (m-3) \log 2 - m > - \log n$$

which can be written as

$$m < C_8 \log n .$$

This completes the proof of Theorem 1.

## 4.4 OPEN PROBLEM

On the basis of Theorem 1 and the corresponding Theorem 1 of Chapter 2, we formulate the following unproven theorem:

Theorem 2. (conjecture)

If (4.1.1)is exact for all polynomials of degree $\leq 2m-1$, $m < n$, with $A$ and $B$ real, and $y$ and $x_i (i=1,2,...,n-1)$ in $(0,\infty)$, then

$$m < K \log n$$

for some constant $K$ independent of $n$.

This conjecture is also offered when the weight function in (4.1.1) is $e^{-x} x^{\alpha}$, $\alpha > -1$.

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